Example 8: Find $\lim_{x \to 2} \frac{3x^2 - 2x + 7}{x^3 + 5x}$ and point out how Theorems 1-C, 1-D, and 1-E are used in the solution.

First we observe that $3x^2 \to 3 \cdot 2 \cdot 2 = 12$ as $x \to 2$. This is by an application of the corollary of Theorem 1-D, with $f_1(x) = 3, f_2(x) = x, f_3(x) = x$. The product rule shows likewise that $-2x \to -2 \cdot 2 = -4$, $x^3 \to 2 \cdot 2 \cdot 2 = 8$, and $5x \to 5 \cdot 2 = 10$ as $x \to 2$. The rule for sums (Theorem 1-C and its corollary) then shows that

$$3x^2 - 2x + 7 \to 12 - 4 + 7 = 15$$

and

$$x^3 + 5x \to 8 + 10 = 18$$

as $x \to 2$. Then, by the theorem for quotients,

$$\lim_{x \to 2} \frac{3x^2 - 2x + 7}{x^3 + 5x} = \frac{15}{18} = \frac{5}{6}.$$

A function which is defined by the quotient of two polynomials in x is called a *rational* function of x. The function in Example 8 is rational.

If the numerator and denominator of a rational function both happen to be 0 when $x = x_0$, this indicates that the numerator and denominator are both divisible by some power of $x - x_0$. Before attempting to find the limit of the function as x approaches x_0, the highest common power of $x - x_0$ should be canceled from numerator and denominator. This is illustrated in the next example.

Example 9: Find $\lim_{x \to 3} \frac{x^3 - x^2 - 9x + 9}{x^2 - x - 6}$. As long as x is neither 3 nor -2 we have

$$\frac{x^3 - x^2 - 9x + 9}{x^2 - x - 6} = \frac{(x - 3)(x + 3)(x - 1)}{(x - 3)(x + 2)} = \frac{(x + 3)(x - 1)}{x + 2}.$$

Since we require $x \neq 3$ in considering the limit as $x \to 3$, we can write

$$\lim_{x \to 3} \frac{x^3 - x^2 - 9x + 9}{x^2 - x - 6} = \lim_{x \to 3} \frac{(x + 3)(x - 1)}{x + 2} = \frac{12}{5}.$$

Continuity

The noun *continuity* and the adjective *continuous* are used in a special technical sense in mathematics to describe a certain quality which a function may or may not possess at a particular value of x. The use of the word "continuous" for this quality is suggested by the everyday use of the word "continuous" to mean "unbroken," or "without interruption." Before giving the exact mathematical definition of continuity, consider an example which illustrates *discontinuity* (i.e., lack of continuity). The "postage function" of Example 4, § 1-6 is discontinuous at each of the points $x = 1, 2, 3, \cdots$, but continuous at all other points x for which it is defined. The discontinuity is expressed by the sudden jump in the value of the function as x passes one of the values $1, 2, 3, \cdots$. On the other hand, the function

To derive the equation of a rectangular hyperbola with the axes as asymptotes, let the foci be at $\left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ and $\left(-\frac{c}{\sqrt{2}}, \frac{-c}{\sqrt{2}}\right)$. Since $a = b$ we have $c^2 = 2a^2$, $2a = c\sqrt{2}$. The definition of the hyperbola is expressed by the equations

$$\sqrt{\left(x + \frac{c}{\sqrt{2}}\right)^2 + \left(y + \frac{c}{\sqrt{2}}\right)^2} - \sqrt{\left(x - \frac{c}{\sqrt{2}}\right)^2 + \left(y - \frac{c}{\sqrt{2}}\right)^2} = \pm c\sqrt{2}.$$

On elimination of the radicals by squaring, just as we did in deriving (2) from (1), we arrive at the equation

$$xy = \frac{c^2}{4}. \tag{11}$$

The definition of a hyperbola makes this type of curve useful in various types of range-finding work. One example is that of locating an enemy artillery piece. Three range-finder listening posts are in contact by telephone. When the gun is fired, each post notes the time at which the shot is heard. By comparing with each other, each pair of posts can determine the difference in the distance from the gun to the two posts. This places the gun on a certain hyperbola with the two posts as foci. Using two different pairs of posts, two hyperbolas are found. The gun is then located graphically at the intersection of the hyperbolas.

Another use of hyperbolas is in blind flying. Two radar beacon stations are used as foci, and it is desired to make a plane fly a course following one branch of a hyperbola with foci at the beacons. Each station sends out a pulse signal which is picked up by the plane and registered on an instrument which shows the distance from the plane to the beacon. The plane then flies so as to maintain a prescribed constant difference in distance from the beacons. In practice one of the beacon signals is usually sent out with a preset delay, so that the plane maintains an *apparently* equal distance from the beacons.

There is a property of the hyperbola corresponding to the so-called "optical properties" of the parabola and ellipse. It is this: The tangent to a hyperbola at a point P bisects the angle between the lines joining P to the two foci. Proof of this is left for an exercise.

EXERCISES

1. Draw the hyperbola in each case. Make a figure like that in Fig. 3-25, or a corresponding one if the foci are on the y-axis. Begin by finding a, b, c and drawing the asymptotes.

culate the length of a typical strip (in terms of x if the width is Δx, in terms of y if the width is Δy), and write down the expression for the area of the rectangle which serves as an approximation to the area of the strip.

(iv) Set up the integral which is the limit of the sum of the expressions of which a typical one was found in step (iii). Observe that if the expression for the area of the rectangle is $F(x)\,\Delta x$, the integral will be

$$\int_a^b F(x)\,dx.$$

The limits of integration are found by examining the figure.

(v) Carry out the integration and find the value of the definite integral.

We give an illustrative example in which the strips are taken parallel to the x-axis.

Example: Find the area enclosed between the parabolas $y^2 = -4(x - 1)$ and $y^2 = -2(x - 2)$.

The curves and a typical strip parallel to the x-axis are shown in Fig. 6-10.

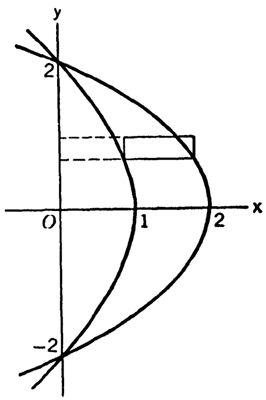

Fig. 6-10

Both parabolas are symmetric with respect to the x-axis, and they open to the left. We solve each equation for x in terms of y:

$$x = \frac{1}{4}(4 - y^2), \qquad x = \frac{1}{2}(4 - y^2).$$

The difference of these two values of x gives the length of a typical rectangle. Hence the area of the rectangle is $\frac{1}{4}(4 - y^2)\,\Delta y$. The area is therefore

$$A = \int_{-2}^{2} \frac{1}{4}(4 - y^2)\,dy.$$

Note the limits of integration. It is evident from symmetry that we may integrate from 0 to 2 and double the result. Thus

$$A = \frac{1}{2}\int_0^2 (4 - y^2)\,dy = \frac{1}{2}\left[4y - \frac{1}{3}y^3\right]_0^2 = \frac{8}{3}.$$

6-6 Three-Dimensional Figures

In this section explanations will be given of the use of rectangular coordinates in space of three dimensions. The primary purpose is to teach the student a few simple things about the description of certain curves and surfaces by equations involving the coordinates x, y, z. Also, there is some illustration of a useful method of drawing diagrams to represent simple solids and surfaces.

A rectangular coordinate system is based on three mutually perpendicular straight lines with a common point of intersection O, called the *origin* of coordinates. The lines are called coordinate axes and each plane determined by two of the axes is called a coordinate plane. A positive direction is assigned along each axis, and each axis is provided with a number scale whose zero point is at O. We shall assume that the same unit of distance is used on each axis. Figure 6-11 shows how the coordinates of a point P are determined. The positive axes are lettered x, y, z, respectively. The x-coordinate of P is defined as the *directed distance* from the yz-plane to the point P. This distance is measured along the perpendicular to the plane, and is positive or negative according as P is on the same side of the yz-plane as the positive or negative portion of the x-axis. In Fig. 6-11 the coordinates of P are the directed distances $OA = x$, $AB = y$, $BP = z$.

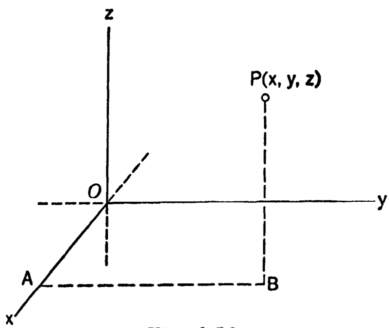

Fig. 6-11

The coordinate system shown in Fig. 6-11 is called *right-handed*. If the labels on the positive x and y axes were exchanged it would be called *left-handed*. We shall always use right-handed systems.

The three coordinate planes divide space into eight regions called *octants*. The one in which all the coordinates are positive is called the *first* octant.

To find the distance $\overline{OP}$ in Fig. 6-11, we use the theorem of Pythagoras twice:

$$\overline{OB}^2 = \overline{OA}^2 + \overline{AB}^2 = x^2 + y^2, \qquad \overline{OP}^2 = \overline{OB}^2 + \overline{BP}^2,$$

and so

$$\overline{OP}^2 = x^2 + y^2 + z^2. \tag{1}$$

From this we see that P lies on a sphere of center O and radius r if and only if

$$x^2 + y^2 + z^2 = r^2. \tag{2}$$

Hence (2) is an equation of this sphere. For many problems relating to such a sphere it is convenient to make a diagram representing merely the first-octant portion of the sphere. Figure 6-12 is such a diagram; on it are

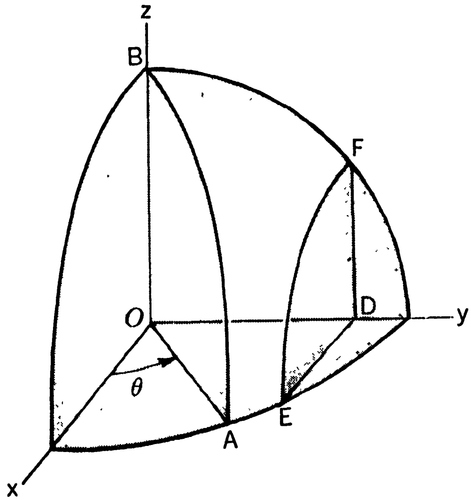

Fig. 6-12

shown the sections of this octant of the sphere by certain planes. The section OAB is made by a plane which passes through the z-axis and makes an acute angle θ with the xz-plane. The arc AB is, of course, a quarter circle of radius r. The section DEF is made by a plane perpendicular to the y-axis. The arc EF is a quarter circle. If $OD = a$, where $0 < a < r$, then $DE = DF = \sqrt{r^2 - a^2}$.

In § 6-7 there is explained a method of finding the volume of a solid by integration. One of the essential steps in the method requires that we be able to find the areas of all sections of the solid made by planes perpendicular to some selected line. When the sections are elementary figures such as triangles, rectangles, or circles, the finding of the areas is often a fairly simple matter.

Example 1: A solid has the following shape: its base is a circle of radius 2, and plane sections of the solid perpendicular to a fixed diameter BB' of the base are isosceles triangles having chords of the circle as their bases. The third vertex of each isosceles triangle lies along one of the lines BC, $B'C$, where C is a point 3 units directly above the center O of the circular base.

To visualize the solid, let the circular base be placed in the xy-plane with the center at the origin and the fixed diameter BB' along the x-axis. Let the point C fall on the positive z-axis. In Fig. 6-13 we show only a quarter of the solid and half of a typical section of the solid by a plane perpendicular to the

all points of the line belong to the surface. Such lines in the surface are called *elements* of the cylinder. In general, a single equation in just two of the three coordinates x, y, z is the equation of a cylinder. If z (for example) is the missing letter, the elements of the cylinder are parallel to the z-axis. In that case the shape of the cylinder is revealed by considering the equation in x and y as the equation of a curve in the xy-plane. All sections of the cylinder by planes parallel to the xy-plane are exactly congruent curves. For instance, the equation $x^2 = 4y$ represents a parabolic cylinder with elements parallel to the z-axis. The cylinder is sym-

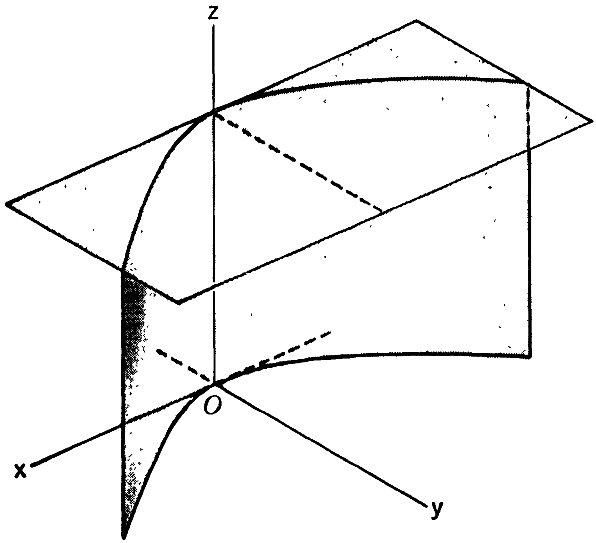

Fig. 6-14

metric with respect to the yz-plane. See Fig. 6-14, in which is shown a part of the cylinder cut off between the xy-plane and a parallel plane.

Example 2: Consider Fig. 6-15. The curve AED is supposed to be a parabola with vertex at D and axis along DO. If $OA = 1$ and $OD = 3$, this parabola is described by the equations

$$y = 0, \qquad 3x^2 = -(z - 3). \tag{4}$$

The curve CGD is also supposed to be part of a parabola with vertex at D and axis along DO. If $OC = 2$, this parabola is described by the equations

$$x = 0, \qquad 3y^2 = -4(z - 3). \tag{5}$$

The equation $3x^2 = -(z - 3)$ by itself represents a parabolic cylinder. Lines AB and EF are segments of elements of this cylinder. Likewise, the equation $3y^2 = -4(z - 3)$ by itself represents a parabolic cylinder, and lines BC and FG are segments of elements of it. These two cylinders intersect in the first

octant along the curve DFB. The figure $EFGH$ is a rectangle in a plane parallel to the xy-plane. If $OH = z$, the dimensions of this rectangle can be found from (4) and (5). Using (4) we have

$$HE = x = \sqrt{(3 - z)/3},$$

and using (5) we have

$$HG = y = 2\sqrt{(3 - z)/3}.$$

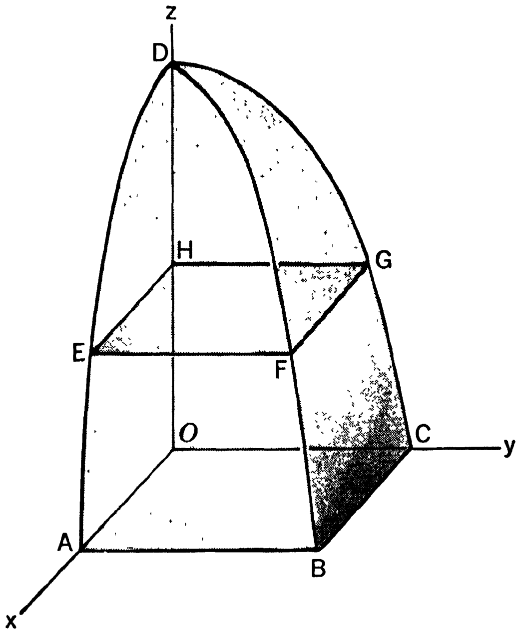

Fig. 6-15

References to Fig. 6-13 and Fig. 6-15 will be made in § 6-7. For a fuller discussion of many topics from analytic geometry of three dimensions, the student may refer to Chapter XVIII.

6-7 Volumes by Slicing

In § 6-1 it was explained how the volume of a solid of revolution can be expressed as a definite integral. In order to achieve this, the solid is cut into thin slices by a series of planes perpendicular to the axis of symmetry of the solid of revolution. This process may be applied to solids of other types. The essential matter for success of the method is that we be able to find the area of each section of the solid made by a plane perpendicular to some fixed line.

Let us consider planes perpendicular to the x-axis, and suppose the

(g) The solid generated by revolving about the x-axis the area bounded by $xy = 1$, $x = 1$, $x = 3$, and $y = 0$;
(h) The solid generated by revolving about the y-axis the area between the two branches of the hyperbola $9x^2 - 16y^2 = 144$ and between the lines $y = 0$, $y = 3$;
(i) The solid generated by revolving about the y-axis the area between the coordinate axes and the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

6. Draw the parabola $Hx^2 = B^2y$, where B and H are positive constants.
(a) Find the volume generated when the area between the parabola and the line $y = H$ is revolved about the y-axis. Compare your answer with the volume of a cylinder of height H and radius of base B.
(b) Find the volume generated when the area between the parabola and the x-axis, $0 \leq x \leq B$, is revolved about the x-axis.

7. A solid has as its base the area bounded by the hyperbola $16x^2 - 9y^2 = 144$ and the line $x = 6$. Every cross section of this solid perpendicular to the x-axis is (a) a square, or (b) an equilateral triangle. Find the volume in each case.

8. Find the volume of the first octant solid of Fig. 6-13, as described in Example 1, § 6-6.

9. Refer to Fig. 6-13 and assume that ADB is a quarter circle. Find the volume of the solid $OABC$ if $OA = a$ and $OC = c$.

10. In Fig. 6-13 let ADB be one quarter of an ellipse, with $OA = a$, $OB = b$, $OC = c$. Find the volume $OABC$.

11. A solid has as its base the triangle cut from the first quadrant by the line $3x + 4y = 12$. Every plane section of the solid perpendicular to the y-axis is a semicircle. Find the volume of the solid.

12. Find the volume generated by revolving about the y-axis the area between $xy = 4$ and $x + y = 5$.

13. The axes of two right circular cylinders, each of radius a, intersect at right angles. Find the volume of the space which is inside of both cylinders.

14. In felling a tree a woodsman first saws halfway through at right angles to the trunk. He then makes a second cut in a plane inclined at an angle θ to the first cut, the two planes meeting in a line which intersects the central axis of the tree. Find the volume of the wedge removed if the tree is assumed to be a cylinder of radius b.

15. A square hole of side 2 inches is cut through a cylindrical post of radius 2 inches. If the axis of the hole intersect the axis of the post at right angles, find the volume cut out (a) assuming that a pair of opposite plane sides of the hole are perpendicular to the axis of the post; (b) assuming that the plane sides of the hole make 45° angles with the axis of the post.

and thus throw (3) into the form

$$Ax + By + C = 0,$$

with A and B prescribed ahead of time. This means that every possible slope is obtainable, and so we get all members of the family in this way. In order to get the particular line which goes through (2, 2), we put $x = y = 2$ in (2). Then

$$4h - 14k = 0, \quad \text{or} \quad 2h - 7k = 0. \tag{4}$$

We do not expect to find unique values for h and k; it is only their ratio which is determined. We can choose any nonzero value of k in (4) and then solve for h. We take $k = 2$ and get $h = 7$; then (3) becomes

$$20x + 27y - 94 = 0.$$

Example 3: The family of all tangents to a given curve is often quite interesting. We shall exhibit the family of all tangents to the parabola $y = x^2$. Let (α, β) be the point of tangency, so that $\beta = \alpha^2$. We shall use α as the parameter in describing the family. From $dy/dx = 2x$ we see that the slope at (α, β) is 2α. Hence the equation of the tangent is

$$y - \alpha^2 = 2\alpha(x - \alpha), \quad \text{or} \quad y = 2\alpha x - \alpha^2.$$

This equation shows, for instance, that the y-intercept of each tangent is the negative of the y-coordinate of the point of tangency. If we wish to find the

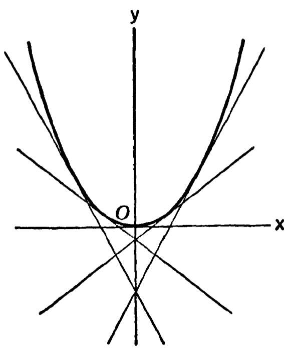

Fig. 7-6

tangent which goes through a particular point (x_0, y_0) in the plane, we try to find α so that $y_0 = 2\alpha x_0 - \alpha^2$. This quadratic has solutions

$$\alpha = \frac{2x_0 \pm \sqrt{4x_0^2 - 4y_0}}{2} = x_0 \pm \sqrt{x_0^2 - y_0}.$$

There are two solutions if $x_0^2 - y_0 > 0$; there is one solution if $x_0^2 - y_0 = 0$; and there is no solution (since α must be real) if $x_0^2 - y_0 < 0$. Observe that $y_0 > x_0^2$ means that the point (x_0, y_0) is inside the parabola. These findings agree with what we expect from looking at Fig. 7-6.

the base a of y, and we write $x = \log_a y$. The laws of exponents become properties of logarithms. If $a^u = A$ and $a^v = B$, then $AB = a^{u+v}$. Hence, since $u = \log_a A$, and so on, we see that

$$\log_a (AB) = \log_a A + \log_a B. \tag{5}$$

The law of exponents in (2) leads to the following law of logarithms:

$$\log_a A^v = v \log_a A. \tag{6}$$

Especially to be noted are the particular facts,

$$\log_a a = 1 \quad \text{and} \quad \log_a 1 = 0. \tag{7}$$

If we wish to study the function $f(x) = \log_a x$, we note that $y = \log_a x$ is equivalent to $a^y = x$. In particular, x must be positive in order that $\log_a x$ may be defined. The appearance of the graph of $y = \log_a x$ can be deduced from the appearance of the graph of $y = a^x$; we must exchange the roles played by x and y. Figure 8-3 shows the graph of $y = \log_a x$ when $a > 1$. In this case $\log_a x$ increases continuously as x increases. The facts corresponding to those in (4) are

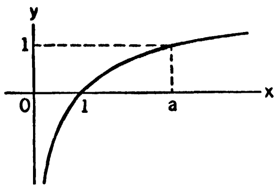

Fig. 8-3

$$\lim_{x \to 0^+} \log_a x = -\infty, \qquad \lim_{x \to +\infty} \log_a x = +\infty. \tag{8}$$

The conception of logarithms was developed near the end of the 16th century. Their earliest use seems to have been mainly for simplifying computations in astronomy; a number of tables were constructed early in the 17th century. The computational usefulness of logarithms (as in high school trigonometry, for instance) is only one aspect of their importance for mathematics. Actually, the logarithm as a function is of very great importance in theoretical work. It is our immediate objective in the next few sections to learn about logarithmic and exponential functions in connection with differentiation and integration.

EXERCISES

1. Find the value of each logarithm.

(a) $\log_2 32$.
(b) $\log_{1/2} 64$.
(c) $\log_3 \dfrac{1}{81}$.
(d) $\log_{9/4} \frac{2}{3}$.
(e) $\log_{0.1} 10{,}000$.
(f) $\log_9 27$.

2. Deduce from (5) that $\log_a \dfrac{A}{B} = \log_a A - \log_a B$; then from this show that $\log_a B^{-1} = -\log_a B$.

8. Newton's law of cooling states that the difference x between the temperature of a body and that of the surrounding air decreases at a rate proportional to this difference. If $x = 100°$ when $t = 0$ and $x = 40°$ when $t = 40$ minutes, find (a) when $x = 70°$; (b) when $x = 16°$; (c) the value of x when $t = 20$.

9. A flywheel spinning about a shaft is slowed down by friction at a rate proportional to the speed of rotation, so that $dp/dt = -kp$, where k is a positive constant and p is the angular velocity of the flywheel. If the initial angular velocity is 1600 revolutions per minute, and if the velocity is halved in 2 minutes, find (a) the angular velocity after t minutes; (b) the time when $p = 100$ revolutions per minute; the number of radians through which the flywheel has turned in t minutes.

10. If $\mu = 0.5$ for a yacht hawser around a wharf post, how many turns of the rope around the post are necessary in order that a man holding the rope can withstand a pull 100 times as great as that of which he is capable?

11. A 60-pound weight is fastened to one end of a rope. The rope goes straight up, over a horizontal spar of circular cross section, and comes straight down to where a man is standing. If the coefficient of friction between rope and spar is 0.35, how heavy is the man if he can just barely lift himself on the rope without raising the 60-pound weight?

12. In formula (5) P is called the *present value.* Solve the equation for P. If the timber on a certain tract will bring \$100 $\cdot\, e^{\sqrt{t}/2}$ when cut t years from the present, for what value of t is the present value of the timber greatest, assuming that interest is compounded continuously at the nominal rate of 5 per cent per year?

13. If a timber tract costs \$1088 to plant and if the cut timber will bring \$400 $\cdot\, e^{\sqrt{t}/2}$ after t years, show that the tract will earn the highest nominal rate of interest upon the initial investment if the timber is cut in 16 years. What is this highest rate? Assume $e = 2.72$, and consider that interest is compounded continuously.

14. A man saves at the constant rate of \$1.00 a day, and invests his money. If one thinks of the savings as going into his account continuously, and if interest is earned at the rate of 4 per cent, compounded continuously, how long will it take the man to accumulate \$10,000? Express his savings x in t years as a function of t.

15. A piece of real estate worth \$20 billion in 1956 is alleged to have been worth \$20 in 1636. What rate of interest, continuously compounded, would yield this same increase in the same time?

16. A room of volume 12,000 cubic feet had the ventilators closed, and the carbon dioxide content of the air in the room was 0.12 per cent (by volume). The ventilators were then opened, and fresh air, with 0.04 per cent carbon dioxide content, was pumped into the room at a fixed rate.
(a) If in 10 minutes the proportion of carbon dioxide was down to 0.06

provided that we can show that this sum does indeed approach a limit as the number n is increased and the greatest length of the individual segments P_0P_1, P_1P_2, $\cdots$ is made to approach zero. In order to show that the sums do approach a limit we must have some rather exact information about the nature of the curve C.

We therefore begin by considering a case of general interest in which we can accomplish this goal. We suppose that C is the graph of $y = f(x)$, where f is a function which has a continuous derivative, and x varies from a to b, where $a < b$. In this case we shall show that the length of C is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2}$$

Hence, to calculate L, we merely work out the value of the integral.

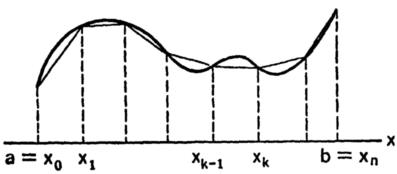

Fig. 11-2

In order to derive the formula (2), consider Fig. 11-2. Here the points $P_0, \cdots, P_n$ along C have been determined by choosing points $x_0, x_1, \cdots, x_n$ along the x-axis from a to b. If $y_k = f(x_k)$, then P_k is the point (x_k, y_k). Now

$$\overline{P_{k-1}P_k} = [(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2]^{1/2}. \tag{3}$$

Since f is continuous, it is evident from (3) that $\overline{P_{k-1}P_k} \to 0$ if $(x_k - x_{k-1}) \to 0$. And certainly the reverse implication is true, because $x_k - x_{k-1} \leq \overline{P_{k-1}P_k}$. Hence, in this case, we have to find the limit of the sum (1) as the greatest of the differences $x_k - x_{k-1}$ approaches zero.

Now let us simplify (3) by using the law of the mean (Theorem 2-C). There is some number X_k between x_{k-1} and x_k such that

$$y_k - y_{k-1} = f(x_k) - f(x_{k-1}) = (x_k - x_{k-1})f'(X_k).$$

We write $\Delta x_k = x_k - x_{k-1}$ for convenience. Then (3) becomes

$$\overline{P_{k-1}P_k} = \{1 + [f'(X_k)]^2\}^{1/2}\, \Delta x_k.$$

But the limit of the sum of all these things is exactly the integral (2), by the definition of the integral.

Example 1: Find the length of the arc of the parabola $4y = x^2$ from $(-2, 1)$ to $(4, 4)$.

Here $dy/dx = x/2$, so the formula is

$$L = \int_{-2}^{4} \sqrt{1 + \frac{x^2}{4}}\, dx = \frac{1}{2}\int_{-2}^{4} \sqrt{4 + x^2}\, dx.$$

$$= \left[\frac{x}{4}\sqrt{4 + x^2} + \log\,(x + \sqrt{4 + x^2})\right]_{-2}^{4}$$

$$= \sqrt{20} + \log\,(4 + \sqrt{20}) + \frac{1}{2}\sqrt{8} - \log\,(-2 + \sqrt{8}).$$

ever. Lest there be a misunderstanding, we state explicitly that the Taylor's series formula (6) is not always valid. Whether or not it is valid in a particular case will depend on the nature of the particular function $f(x)$ and on the particular values of x and a.

For the purpose of familiarizing the student with Taylor's series we shall give some examples and exercises in which the primary purpose is to obtain the Taylor's series for various functions. For this purpose the emphasis will be on the actual calculation of the successive derivatives of $f(x)$ and their evaluation at the point $x = a$. *It will be assumed without proof that in the problems of this kind which are given in this book, the Taylor's series formula* (6) *is actually true whenever the series is convergent.*

Example 4: The following four formulas are instances of Taylor's series with $a = 0$. They are valid for all values of x.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{11}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{12}$$

$$\frac{1}{2}(e^x + e^{-x}) = \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \tag{13}$$

$$\frac{1}{2}(e^x - e^{-x}) = \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \tag{14}$$

The beginning of the series (11) was obtained in Example 2. The derivatives of $\sin x$ repeat in groups of four, so that if $f(x) = \sin x$, we get

$$\begin{aligned} f(0) &= f^{(4)}(0) = f^{(8)}(0) = \cdots = 0 \\ f'(0) &= f^{(5)}(0) = f^{(9)}(0) = \cdots = 1 \\ f''(0) &= f^{(6)}(0) = f^{(10)}(0) = \cdots = 0 \\ f^{(3)}(0) &= f^{(7)}(0) = f^{(11)}(0) = \cdots = -1. \end{aligned}$$

Thus the series (11) contains only the odd powers of x, and the signs on these terms are alternately plus and minus. The general term of (11) can be displayed as

$$(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}, \qquad n = 1, 2, 3, \cdots. \tag{15}$$

Discussion of the series (12), (13) and (14) is left for the Exercises.

In Taylor's series (6) the "general" term is

$$\frac{1}{n!} f^{(n)}(a)(x-a)^n.$$

This formula gives the initial term $f(a)$ provided we follow the conventions that $0! = 1$ and $f^{(0)}(a) = f(a)$. These are standard conventions, and we shall adhere to them.

(b) $\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix}$. (e) $\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$.

(c) $\begin{vmatrix} 2 & 4 & 1 \\ 3 & 5 & 2 \\ 6 & 12 & 3 \end{vmatrix}$. (f) $\begin{vmatrix} -6 & 15 & 3 \\ 5 & -4 & 2 \\ 3 & 1 & 3 \end{vmatrix}$.

5. Explain on the basis of theorems in this section, why each of the following pairs of determinants are equal.

(a) $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 9 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{vmatrix}$.

(b) $\begin{vmatrix} 1 & 3 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & -3 & 4 \\ 4 & -2 & 6 \end{vmatrix}$.

(c) $\begin{vmatrix} 2 & 4 & 1 \\ 3 & 5 & 2 \\ 6 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 4 & -3 \\ 3 & 5 & -4 \\ 6 & 1 & -6 \end{vmatrix}$.

6. (a) See if you can discover the linear dependence of the columns which insures that

$$\begin{vmatrix} 4 & 5 & -2 \\ 7 & -4 & 5 \\ 7 & 2 & 1 \end{vmatrix} = 0.$$

(b) Can you discover also the linear dependence of the rows?

7. Calculate the value of each determinant by methods analogous to that of Example 3.

(a) $\begin{vmatrix} 10 & 15 & 20 \\ 12 & 12 & 32 \\ 2 & 3 & 12 \end{vmatrix}$. (c) $\begin{vmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{vmatrix}$.

(b) $\begin{vmatrix} 3 & 2 & 1 \\ 3 & -3 & 2 \\ 10 & 1 & 7 \end{vmatrix}$. (d) $\begin{vmatrix} 2 & 4 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$.

8. (a) Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix},$$

curve in the xy-plane. It is called a *level curve* of the function. If we know the level curves for various values of k, we can obtain a very good idea of what the function is like. The representation of a function by drawing level curves of it is based on the same idea as that which is used in representing the configuration of the land surface in a certain region by a topographical map of the region.

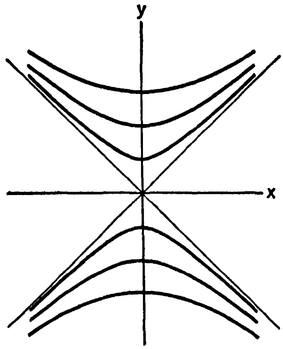

Fig. 19-1

Example 5: If $f(x, y) = \sqrt{y^2 - x^2}$, the level curves are defined by $\sqrt{y^2 - x^2} = k$.

The only admissible values of k are positive or zero. For $k = 0$ we get $y^2 - x^2 = 0$, which represents the two lines $y = \pm x$. For $k > 0$ the level curves are rectangular hyperbolas as shown in Fig. 19-1. The curves shown correspond to $k = 1, 2, 3$. The graph of $z = \sqrt{y^2 - x^2}$ is the portion of the conical surface $z^2 = y^2 - x^2$, or $x^2 + z^2 = y^2$, on which $z \geq 0$. It is a right circular cone with axis along the y-axis. The level curve $\sqrt{y^2 - x^2} = k$ is just like the curve in which the plane $z = k$ intersects the cone.

In the general case, the level curves $f(x, y) = k$ are just like the curves in which the various planes $z = k$ intersect the surface defined by $z = f(x, y)$.

Level Surfaces

If f is a function of three variables x, y, z, and if we write $w = f(x, y, z)$, a graphical representation of the function can be made by talking about points (x, y, z, w) in space of four dimensions. But physical intuition about functions of three variables may be better served by using the notion of a *level surface*. For a given constant k the locus of points (x, y, z) in three-dimensional space such that $f(x, y, z) = k$ may be a surface. If so, we call it a *level surface*. By visualizing the various level surfaces, we can form an idea of the nature of the function.

Example 6: Let $f(x, y, z) = \dfrac{x^2}{25} + \dfrac{y^2}{16} + \dfrac{z^2}{9}$.

Here the admissible values of k are those for which $k \geq 0$. If $k = 0$, there is no level surface; the locus $f(x, y, z) = 0$ is the single point $(0, 0, 0)$. If $k > 0$, the level surface is an ellipsoid. All these ellipsoids have the same center and the same axes of symmetry. As we go out away from the origin, the values of f increase. As we shall see later, the direction of most rapid increase at a point is the direction perpendicular at that point to the ellipsoid which is the level surface.

and $c \leq y \leq d$, where a, b, c, d are numbers such that $a < b$ and $c < d$. Some or all of these numbers may be negative. Let f be a function of x and y which is defined at each point of R. The function might perhaps be defined at some points not in R, but we ignore this and regard R as the domain of definition of f. Ordinarily we consider situations in which f is continuous at each point of R, but it would do no harm to have certain types of discontinuous behavior of f. However, we shall not attempt to describe precisely what might be permissible in this respect, and we shall for the present assume that f is continuous at each point of R.

Definition of a Double Integral

Now let R be divided into a number of smaller rectangles in the following manner: Choose numbers $x_0, x_1, \cdots, x_m$ and $y_0, y_1, \cdots, y_n$ so that

$$a = x_0 < x_1 < \cdots < x_m = b, \qquad c = y_0 < y_1 < \cdots < y_n = d,$$

and draw the various lines $x = x_j$, $y = y_k$, so that R is divided into mn rectangles, as shown in Fig. 20-1. We write $\Delta x_j = x_j - x_{j-1}$, $\Delta y_k = y_k - y_{k-1}$. Let R_{jk} be the rectangle with sides $x = x_{j-1}$, $x = x_j$, $y = y_{k-1}$, $y = y_k$ (shaded in Fig. 20-1). In R_{jk} choose any point $x = s_j$, $y = t_k$, and form the product $f(s_j, t_k)\,\Delta x_j\,\Delta y_k$, which is the value of f at (s_j, t_k) multiplied by the area of R_{jk}. Then form the sum of all these products; this sum can be written in the form

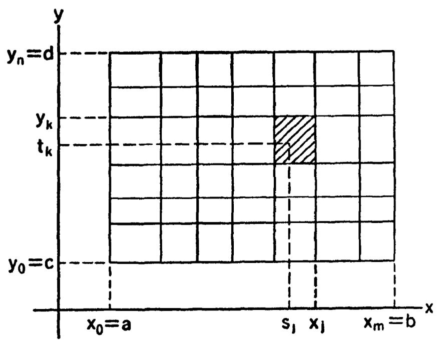

Fig. 20-1

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_k)\,\Delta x_j\,\Delta y_k \tag{2}$$

by using a summation symbol notation. Now consider what happens as the maximum of all the numbers $\Delta x_1, \cdots, \Delta x_m, \Delta y_1, \cdots, \Delta y_n$ is made to approach zero. (This will, of course, force m and n to increase indefinitely.) It turns out that the sums (2) approach a definite limit, and this limit is called the value of the double integral of f over the region R. The value of the integral is indicated by either of the notations in (1). The meaning of the sums approaching their limit is that the absolute value

$$\left|\iint_R f(x, y)\,dA - \sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_k)\,\Delta x_j\,\Delta y_k\right| \tag{3}$$

can be made as small as we please, simply by making the greatest of the Δx_j's and Δy_k's sufficiently small. Apart from this condition on the Δx_j's

and Δy_k's it does not matter how the points x_j and y_k are spaced, nor how the points (s_j, t_k) are chosen in R_{jk}. The dA notation in (3) is suggested by using ΔA_{jk} instead of $\Delta x_j\, \Delta y_k$ to denote the area of R_{jk}; the notation is purely conventional, following historical tradition, for dA is not the differential of a function.

The fact that the sums (2) do approach a limit can be proved as a consequence of the continuity of the function f. This proof is given in more advanced textbooks on calculus.

It is necessary to define double integrals over regions of somewhat arbitrary shape, as well as over rectangles. The procedure is much as before, but there are some differences, owing to the fact that if the boundaries of the region are curved, the region cannot be exactly filled out by small rectangles.

Suppose now that R is a region which is bounded by one or more closed curves. We suppose that the curves are composed of a finite number of simple arcs, each defined either by specifying y as a function of x on some interval of the x-axis, or the corresponding situation with the roles of x and y reversed; these functions shall be continuous. The simplest typical case would be where there is just one closed circuit (e.g., a triangle, a polygon, an ellipse, or a semicircle and a diameter) forming the boundary of R. But R might also be such a thing as a circular region with a square hole cut out of it. Let two sets of lines be drawn, one set parallel to the x-axis, the other set parallel to the y-axis. The spacing of the lines need not be regular, but the lines should be close enough together so that the rectangles formed (see Fig. 20-2) are small in comparison with the size of R. The network of lines forms what we call a *rectangular partition;* an individual rectangle in the partition is called a *cell.* Some of the cells will lie entirely in the region R; other cells will contain points not in R. For our purposes we retain only those cells which do not in any way extend outside of R. We then number the retained cells in some arbitrary order. If there are N cells, let their areas be $\Delta A_1, \cdots, \Delta A_N$. Now suppose that f is a function which is continuous at each point of R. Choose any point (x_i, y_i) in the ith cell, and form the sum

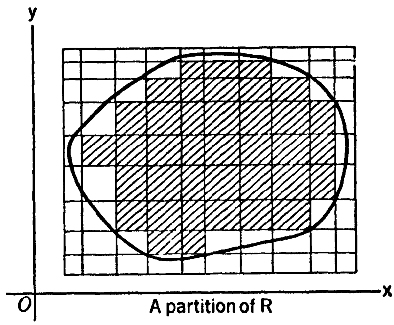

Fig. 20-2

$$\sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i. \tag{4}$$

be $\iint_R f(x, y)\, dA$, and suppose that $f(x, y)$ is transformed into $F(r, \theta)$ when we set $x = r \cos \theta$, $y = r \sin \theta$. For example, if $f(x, y) = xy^2$, this becomes $r \cos \theta\, (r \sin \theta)^2 = r^3 \cos \theta \sin^2 \theta$, which is $F(r, \theta)$.

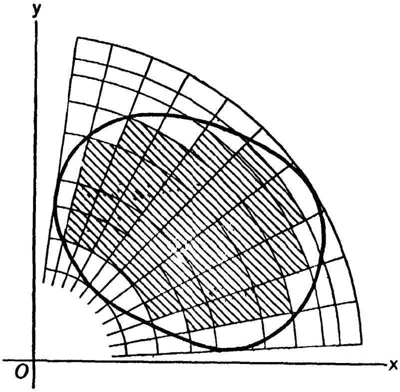

Fig. 20-10

Now let us form a polar coordinate partition of the plane by a series of circles with center at the origin and a series of rays emanating from O (see Fig. 20-10). This partition of the plane forms cells which are rather like rectangles. We now proceed with a process much like that described in connection with Fig. 20-2. We select those cells which belong completely to the region R (these cells are shaded in Fig. 20-10) and number them consecutively in some order. Suppose there are N such cells. If ΔA_k is the area of the kth cell, and if (x_k, y_k) is any point of the cell, it seems reasonable to suppose that

$$\lim \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA, \tag{1}$$

the limit being taken in the sense that the partition is made finer and the cell size approaches zero. The truth of (1) is plausible if we think of the interpretation of the double integral as a volume; it is also plausible in the case when $f(x, y)$ is a density function and the integral is interpreted as the total mass of a lamina. We shall not attempt a formal proof of (1), but we shall use the result as basic in our argument. The whole subject can be treated rigorously in the theory of transformation of multiple integrals—a subject which is dealt with in books on advanced calculus.

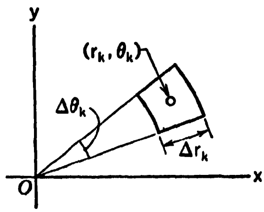

Fig. 20-11

The next step is to express $f(x_k, y_k)\, \Delta A_k$ in terms of polar coordinates. Consider the kth cell, as shown in Fig. 20-11. Let the polar coordinates of (x_k, y_k) be (r_k, θ_k), and let us choose this point in the special position midway between the two circular arcs and midway between the two circular rays. Then the two circular arcs have radii

$$r = r_k - \tfrac{1}{2}\Delta r_k, \qquad r = r_k + \tfrac{1}{2}\Delta r_k.$$

The area of the cell is easily worked out by elementary geometry, starting from the formula for the area of a circular sector in terms of its radius and

inertia of the given lamina about the axis L is $1/R^2$. Because of this the ellipse is called the *ellipse of inertia* for the given lamina, relative to the point O. The result just stated shows that I is smallest when L coincides with the major axis of the ellipse of inertia. The axes of symmetry of the ellipse are called *principal axes of inertia* of the lamina (relative to O).

11. Compute the product of inertia (9) for the homogeneous lamina defined by $x^2 + y^2 \leq a^2$, $y \geq 0$, $(x - a)^2 + y^2 \leq a^2$.

12. Get the equation of the ellipse of inertia relative to O for the lamina of density $\sigma = (x + y)^2$ occupying the region $x^2 + y^2 \leq a^2$. What are the principal axes of inertia?

13. Among all axes parallel to a given line, about which one is the moment of inertia of a given mass system the least?

14. Suppose $f(x) \geq 0$ when $a \leq x \leq b$, f being a continuous function. Consider the volume generated when the area between the curve $z = f(x)$ and the x-axis in the xz-plane, from $x = a$ to $x = b$, is revolved around the z-axis. Show that this volume is given by the double integral $\iint\limits_R f(\sqrt{x^2 + y^2})\, dx\, dy$, where R is the region between the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$ in the xy-plane. Express the double integral as an iterated integral in polar coordinates and show that the result is in agreement with the shell method of finding volumes of solids of revolution, in § 11-2.

20-4 Mass Systems and Newton's Law

The present section is a digression from the subject of multiple integrals. We shall discuss the way in which Newton's second law of motion is applied to the study of the motion of a mass system. The discussion brings out the importance of the concept of center of mass and also the importance of the concept of moment of inertia.

We begin with the consideration of a rigid mass system which is rotating about a fixed axis. When we describe the system as rigid we mean that if we fix our attention on any two points in the mass system, the distance between these two points does not change as the whole system moves. As a consequence of the rigidity, when the system rotates, any particular point of the mass system describes a circular path about the axis of rotation, *and all points move with the same angular velocity*. A mass particle m_k at distance r_k from the axis moves with speed $v_k = \omega r_k$, where ω is the angular velocity. Hence its kinetic energy is

$$\tfrac{1}{2}m_k v_k^2 = \tfrac{1}{2}m_k r_k^2 \omega^2.$$

If the system consists of n particles, the total kinetic energy is

$$\tfrac{1}{2}\sum_{k=1}^{n} m_k v_k^2 = \tfrac{1}{2}\omega^2 \sum_{k=1}^{n} m_k r_k^2 = \tfrac{1}{2}I\omega^2,$$

cussion of how to solve various kinds of differential equations. It is clear how one can go on to define differential equations of third order, fourth order, and so on.

Examples:

$$3y + x\frac{dy}{dx} = 9xy^2 \qquad \text{(first order);}$$

$$x\frac{d^2y}{dx^2} + (3x^2 + 1)\frac{dy}{dx} = x^3 \qquad \text{(second order);}$$

$$y^2 = x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \text{(first order);}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 \qquad \text{(second order).}$$

EXERCISES

1. Suppose $f(x) = 0$ if $0 < x < 1$, $f(x) = 1$ if $1 \le x < 2$. Show that it is impossible to find a function F defined and differentiable for *all* x such that $0 < x < 2$ and such that $F'(x) = f(x)$ for all these values of x. Begin by showing what $F(x)$ must be like for $0 < x < 1$ and for $1 < x < 2$, assuming that an F of the required sort does exist. Where does the impossibility show up?

2. (a) Exhibit an integral formula which shows what F must be like if $y = F(x)$ and $d^2y/dx^2 = f(x)$ when $a \le x \le b$, given that f is continuous when $a \le x \le b$.
(b) Select the particular F which meets the condition of (a) if in addition $F'(a) = A$ and $F(a) = B$.
(c) Select the particular F which meets the condition of (a) if in addition $F(a) = F(b) = 0$.

21-2 First-Order Equations with Variables Separable

If f is a function of two variables, the equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is a rather general type of differential equation of the first order. By a solution of this equation we shall mean a differentiable function F of x such that

$$F'(x) = f[x, F(x)] \tag{2}$$

for all x on some interval. There is no a priori reason why such a solution should exist, but one may impose conditions on f which suffice to guarantee the existence of solutions.

We shall proceed by assuming a rather special condition on the function f. We assume that f can be expressed as the quotient of a continuous func-

ciated with the point. Now, if there is a curve $y = F(x)$ in R such that at each of its points it is tangent to the direction-element associated with the point, it is clear that $F'(x) = f[x, F(x)]$, and hence that $y = F(x)$ is a solution of the differential equation (1). The curve $y = F(x)$ is then called an *integral curve* of the differential equation.

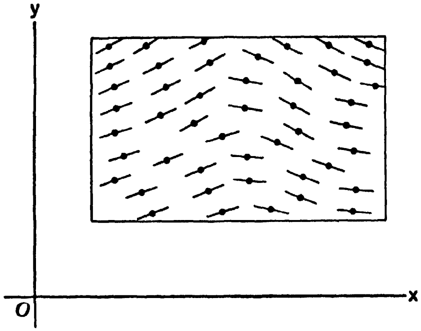

Fig. 21-2

Suppose now that we have a one-parameter family of smooth curves coursing through a rectangle R, as shown in Fig. 21-3. We suppose that through any given point of R there passes exactly one curve of the family, and that no curve ever has a tangent (at a point in R) parallel to the y-axis. Then, at each point (x, y) in R there is a unique slope of the curve through that point. This defines a function f: $f(x, y) =$ slope at (x, y) of the curve which passes through (x, y). Then each curve of the family is an integral curve of the differential equation $y' = f(x, y)$, and a direction-field can be constructed by drawing segments of lines tangent to the curves of the family.

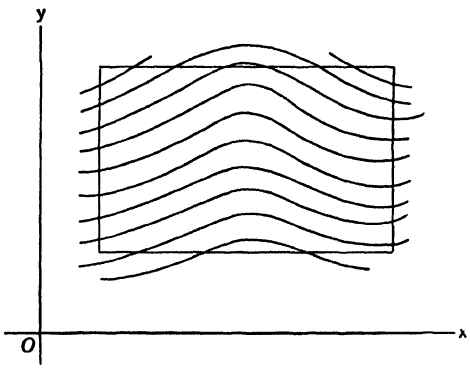

Fig. 21-3

If we merely have the differential equation $y' = f(x, y)$, and we imagine the corresponding direction-field to have been constructed, it is natural to speculate as to whether there really does exist a family of curves having the direction-elements as tangents. In the theory of differential equations, at a more advanced level than our present one, it is shown that, if certain assumptions are made about f, one can prove the existence of a

of the motion. See Fig. 21-6. Show that the relation between v and ϕ is given by the differential equation

$$\frac{1}{v}\frac{dv}{d\phi} = \frac{kv}{g}\sec\phi + \tan\phi.$$

Hence, show that

$$v = \frac{g\sec\phi}{Cg - k\tan\phi},$$

where C is a constant depending on the initial conditions. You will need to use results from Chapter XIII, especially as regards tangential and normal components of acceleration.

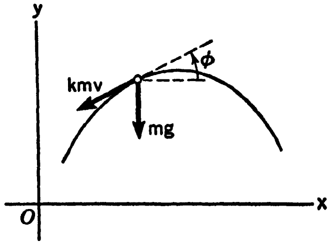

Fig. 21-6

11. (a) On the assumption that air resistance is proportional to the square of the speed, the velocity of an object of mass m falling freely through the atmosphere is governed by the equation $m\dfrac{dv}{dt} = mg - kv^2$, where k is a constant (v positive for downward motion). Solve this equation, assuming initially $v = 0$, $t = 0$. Show that when t is very large, v is approximately $\sqrt{mg/k}$.

(b) In a scientific test a man jumped from an airplane and fell 29,300 feet before opening his parachute. His total weight, with equipment, was 285 pounds. Instruments showed that he reached a limiting velocity of 230 miles per hour. Using the differential equation from (a), find the value of k, and calculate the number of seconds required for the man to attain 99 per cent of his limiting velocity.

12. The velocity of a small lead shot of mass m falling vertically through water obeys the law

$$\frac{dv}{dt} + \frac{a}{m}v = \left(1 - \frac{1}{\rho}\right)g,$$

where $a = 1.69 \times 10^{-2}$, $g = 980$, and ρ is the density of lead. Units are those of the cgs. system. Calculate the limiting velocity of the shot, and the time required to attain half this velocity, starting from rest. Assume $\rho = 11$ and consider the shot to be a sphere of radius 0.05 centimeter.

13. (a) Consider a flexible cord or chain hanging over a horizontal circular cylinder of radius b, with μ the coefficient of friction between the cord and the cylinder. Let σ be the linear density (constant) of the cord. Use a diagram somewhat like Fig. 8-9 in § 8-6, but take into account the weight, $g\sigma b\,\Delta\theta$, of the segment of the cord corresponding to $\Delta\theta$. Let $\theta = 0$ be the horizontal direction. If the cord is on the point of slipping in the direction of increasing θ, show that the tension T is determined by the differential equation

$$\frac{dT}{d\theta} - \mu T = \sigma g b(\mu\sin\theta + \cos\theta).$$